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CITATION:

Tsuchiya, Takuya. Numerical Verification of Simple Bifurcation Points.  
数理解析研究所講究録 1993, 831: 129-140

ISSUE DATE:

1993-04

URL:

<http://hdl.handle.net/2433/83367>

RIGHT:

## Numerical Verification of Simple Bifurcation Points

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**Abstract.** Nonlinear boundary value problems (NBVPs in abbreviation) with parameters are called parametrized nonlinear boundary value problems. This paper studies numerical verification of simple bifurcation points of parametrized NBVPs defined on one-dimensional bounded intervals. Around simple bifurcation points the original problem is extended so that the extended problem has an invertible Fréchet derivative. Then, the usual procedure of numerical verification of solutions can be applied to the extended problem. A numerical example is given.

**Key words.** parametrized nonlinear boundary value problems, numerical verification of solutions, simple bifurcation points

**AMS(MOS) subject classifications.** 65L10, 65L99

**Abbreviated title.** Numerical Verification

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<sup>\*\*</sup> Partially supported by Saneyoshi Scholarship Foundation.

## 1. Introduction.

For the past several years a theory for numerical verification of solutions of differential equations has been developed [N1-5], [TN], [WN]. By the theory the existence of *exact* solutions of differential equations are verified on computers by certain procedures in finite steps.

Let  $\Lambda \subset \mathbb{R}$  be a bounded interval for a parameter. Here we deal with the following nonlinear two-point boundary value problem with a parameter  $\lambda \in \Lambda$  on the bounded interval  $J := (a, b)$ :

$$(1.1) \quad \begin{cases} -u'' = f(\lambda, x, u) & \text{in } J, \\ u(a) = u(b) = 0, \end{cases}$$

where  $f : \Lambda \times J \times \mathbb{R} \rightarrow \mathbb{R}$  is a given smooth function. Since (1.1) has the parameter  $\lambda$ , the set of the solutions of (1.1) would form one dimensional curves. There, however, may exist singular points on the curves. For example, a solution curve might fold (the folding point is call a **turning point**), or several solution curves might intersect at one point (the intersecting point is called a **bifurcation point**).

Let  $(\lambda, u)$  be a solution of (1.1). The above singularities occur when the following eigenvalue problem has the eigenvalue  $\mu = 0$ :

$$(1.2) \quad L\psi = \mu\psi,$$

where the differential operator  $L$  is defined by

$$L\psi := -\psi'' - f_y(\lambda, x, u)\psi,$$

and  $f_y(\lambda, x, y)$  denotes the derivative of  $f$  with respect to  $y$ . More precisely, if  $\mu = 0$  is *not* an eigenvalue of (1.2), by the implicit function theorem, there exists a unique solution curve around  $(\lambda, u)$ , and it is parametrized by  $\lambda$ . Such a solution curve is called a **regular branch**. On regular branches the usual procedure of numerical verification of solutions of (1.1) can be applied.

Suppose that (1.2) has the eigenvalue 0. Then, we have some singularities there; we may have a turning point or, even worse, a bifurcatin point. In [TN] the case of turning points was considered.

In this paper we consider the case of bifurcation points. The difficulty of bifurcation points is as follows. A bifurcation point itself is not only very difficult to compute, but also very instable under perturbation: it may be disappeared by rounding errors or

discretizations. Such a destroyed bifurcation is usually called a **numerical imperfect bifurcation**.

Our goal is to establish a new procedure for numerical verification of bifurcation points. The main idea is as follows: In [W] the original equation is extended around a simple bifurcation points so that the extended equation has an invertible Fréchet derivative. Then a straightforward modification of the usual numerical verification procedure works well at a bifurcation point.

In the last section a numerical examples is given.

## 2. Parametrized NBVP and Simple Bifurcation Points.

As is stated in Section 1, we consider the two-point boundary value problem

$$(2.1) \quad \begin{cases} -u'' = f(\lambda, x, u) & \text{in } J, \\ u(a) = u(b) = 0, \end{cases}$$

where  $J := (a, b) \subset \mathbb{R}$  is a bounded interval, and  $\lambda \in \Lambda \subset \mathbb{R}$  is a parameter.

Let  $H_0^1(J)$ ,  $H^{-1}(J)$ , etc. are the usual Sobolev spaces. In notation we omit '(J)', whenever there is no danger of confusion. The weak form of (2.1) is written as

$$(2.2) \quad \text{Find } u \in H_0^1 \text{ such that } (u', v') = (f(\lambda, x, u), v), \quad \text{for } \forall v \in H_0^1,$$

where  $(\cdot, \cdot)$  is the inner product of  $L^2$  defined by  $(g, h) := \int_J gh dx$  for  $g, h \in L^2$ . Now, define the operators  $L : \Lambda \times H_0^1 \rightarrow H^{-1}$  and  $F : \Lambda \times H_0^1 \rightarrow L^2 \subset H^{-1}$  by, for  $(\lambda, u) \in \Lambda \times H_0^1$ ,

$$(2.3) \quad \langle L(\lambda, u), v \rangle := \int_J u'v' dx, \quad \forall v \in H_0^1,$$

$$(2.4) \quad \langle F(\lambda, u), v \rangle := \int_J f(\lambda, x, u)v dx, \quad \forall v \in H_0^1,$$

where  $\langle \cdot, \cdot \rangle$  is the duality pair of  $H^{-1}$  and  $H_0^1$ . Since the inclusion  $\iota : L^2 \hookrightarrow H^{-1}$  is compact, the operator  $L - F : \Lambda \times H_0^1 \rightarrow H^{-1}$  is a Fredholm operator of index 1.

For  $F$  to be smooth, we suppose the following assumption holds:

A function  $\psi : \Lambda \times J \times \mathbb{R} \rightarrow \mathbb{R}$  is called **Carathéodory continuous** if  $\psi$  satisfies the following conditions: for  $(\lambda, x, y) \in \Lambda \times J \times \mathbb{R}$ ,

$$\begin{cases} \psi(\lambda, x, y) \text{ is continuous with respect to } \lambda \text{ and } y \text{ for almost all } x, \\ \psi(\lambda, x, y) \text{ is Lebesgue measurable with respect to } x \text{ for all } \lambda \text{ and } y. \end{cases}$$

If  $\psi(\lambda, x, y)$  is Carathéodory continuous,  $\psi(\lambda, x, u(x))$  is Lebesgue measurable with respect to  $x$  for any Lebesgue measurable function  $u$ .

Let  $\alpha = (\alpha_1, \alpha_2)$  be usual multiple index with respect to  $\lambda$  and  $y$ . That is, for  $\alpha = (\alpha_1, \alpha_2)$ ,  $D^\alpha f(\lambda, x, y)$  means  $\frac{\partial^{|\alpha|}}{\partial \lambda^{\alpha_1} \partial y^{\alpha_2}} f(\lambda, x, y)$ .

Let  $d \geq 1$  be an integer. For  $\alpha$ ,  $|\alpha| \leq d$ , we define the map  $F^\alpha(\lambda, u)$  for  $(\lambda, u) \in \Lambda \times H_0^1$  by

$$(2.5) \quad F^\alpha(\lambda, u)(x) := D^\alpha f(\lambda, x, u(x)).$$

We then assume that

**Assumption 2.1.** Let  $d \geq 2$ . For all  $\alpha$ ,  $|\alpha| \leq d$ , we suppose that

- (1) For almost all  $x \in J$ ,  $D^\alpha f(\lambda, x, y)$  exists at any  $(\lambda, y) \in \Lambda \times \mathbb{R}$ , and that is Carathéodory continuous.
- (2) The mapping  $F^\alpha$  defined by (2.5) is a continuous operator from  $\Lambda \times H_0^1$  to  $L^2$ , and the image  $F^\alpha(U)$  of any bounded subset  $U \subset \Lambda \times H_0^1$  is bounded.  $\triangleleft$

Assumption 2.1 is satisfied if  $f : \Lambda \times J \times \mathbb{R} \rightarrow \mathbb{R}$  is, for instance,  $C^d$  function.

**Lemma 2.2.** Suppose that Assumption 2.1 holds. Then, the operator  $F : \Lambda \times H_0^1 \rightarrow H^{-1}$  is of  $C^d$  class, and its partial derivatives are written as

$$\begin{aligned} \langle D_u F(\lambda, u)\psi, v \rangle &= \int_J f_y(\lambda, x, u(x)) \psi v dx, \\ \langle D_\lambda F(\lambda, u)\eta, v \rangle &= \eta \int_J f_\lambda(\lambda, x, u(x)) v dx, \end{aligned}$$

for  $\psi, v \in H_0^1$ , and  $\eta \in \mathbb{R}$ .  $\triangleleft$

Now, suppose that  $(\lambda_0, u_0) \in \Lambda \times H_0^1$  satisfies the following:

$$(2.6) \quad \begin{cases} (L - F)(\lambda_0, u_0) = 0 \in H^{-1}, \\ \dim \text{Ker} D_u(L - F)(\lambda_0, u_0) = 1, \\ D_\lambda(L - F)(\lambda_0, u_0) \in \text{Im} D_u(L - F)(\lambda_0, u_0). \end{cases}$$

We denote  $D_u(L - F)(\lambda_0, u_0)$ ,  $D_\lambda(L - F)(\lambda_0, u_0)$  by  $D_u(L - F)^0$ ,  $D_\lambda(L - F)^0$ , respectively. Since  $L - F$  is a nonlinear Fredholm operator of index 1 and (2.6), we know that  $\dim \text{Ker} D(L - F)^0 = 2$  and  $\dim \text{Im} D(L - F)^0 = 1$ . We also find that  $H_0^1$  can be decomposed into subspaces

$$H_0^1 = V_0 \oplus \text{Ker} D_u(L - F)^0,$$

and  $D_u(L - F)^0|_{V_0}$  is an isomorphism from  $V_0$  to  $\text{Im}D_u(L - F)^0$ . Let  $P$  be the inverse operator of  $D_u(L - F)^0|_{V_0}$ . It follows from (2.6) that there exists  $\phi_0 \in H_0^1$  and  $\psi_0 \in H_0^1$  such that

$$(2.7) \quad \begin{cases} D_u(L - F)^0 \phi_0 = 0, & \|\phi_0\|_{H_0^1} = 1, \\ \text{Ker}D_u(L - F)^0 = \text{span}\{\phi_0\}, \\ \text{Im}D_u(L - F)^0 = \{v \in H^{-1} \mid \langle v, \psi_0 \rangle = 0\}. \end{cases}$$

Then we assume that

$$(2.8) \quad B_0^2 - A_0 C_0 > 0,$$

where  $A_0$ ,  $B_0$ , and  $C_0$  are defined by

$$A_0 := \langle D_{uu}^2(L - F)^0 \phi_0^2, \psi_0 \rangle,$$

$$B_0 := \langle D_{\lambda u}^2(L - F)^0 \phi_0 + D_{uu}^2(L - F)^0(\phi_0, -PD_\lambda^0), \psi_0 \rangle,$$

$$C_0 := \langle D_{\lambda\lambda}^2(L - F)^0 + 2D_{\lambda u}^2(L - F)^0(-PD_\lambda^0) + D_{uu}^2(L - F)^0(-PD_\lambda^0)^2, \psi_0 \rangle,$$

and  $PD_\lambda^0 := P(D_\lambda(L - F)^0)$ .

By the Morse Theorem we have the following theorem:

**Theorem 2.3.** *Suppose that Assumption 2.1 holds for  $d \geq 2$ . Assume also that (2.6) and (2.8) are satisfied. Then, there exists an open set of  $(\lambda_0, u_0)$  such that the solutions of*

$$(L - F)(\lambda, u) = 0$$

*consist in two  $C^{d-2}$  branches which intersect transversally at  $(\lambda_0, u_0)$ .*

A point at which (2.6) and (2.8) are satisfied is called a **simple bifurcation point**.

To resolve the singularity at a simple bifurcation point, Weber presented in [W] an extended operator which has an invertible Fréchet derivative at the simple bifurcation point. Let  $X$  and  $Y$  be Banach spaces defined by

$$(2.9) \quad X := H_0^1 \times H_0^1 \times H_0^1 \times \mathbb{R} \times \mathbb{R}, \quad Y := H^{-1} \times H^{-1} \times H^{-1} \times \mathbb{R} \times \mathbb{R},$$

with the norms

$$\begin{aligned} \|(x, y, z, a, b)\|_X &:= \max\{\|x\|_{H_0^1}, \|y\|_{H_0^1}, \|z\|_{H_0^1}, |a|, |b|\}, \\ \|(p, q, r, c, d)\|_Y &:= \max\{\|p\|_{H^{-1}}, \|q\|_{H^{-1}}, \|r\|_{H^{-1}}, |c|, |d|\}, \end{aligned}$$

for  $(x, y, z, a, b) \in X$  and  $(p, q, r, c, d) \in Y$ . Let  $X_0 := (H_0^1)^3 \times \Lambda \times \mathbb{R} \subset X$ . Suppose that  $(\lambda_0, u_0) \in \Lambda \times H_0^1$  is a simple bifurcation point of the equation  $L - F = 0$ . Then we define an extended operator  $G : X_0 \rightarrow Y$  by

$$(2.10) \quad H(u, r, s, \lambda, \mu) := \begin{pmatrix} (L - F)(\lambda, u) + \mu\phi \\ D_u(L - F)(\lambda, u)r \\ D_u(L - F)(\lambda, u)s + D_\lambda(L - F)(\lambda, u) \\ (r, r) - 1 \\ (r, s) \end{pmatrix},$$

where  $(\cdot, \cdot)$  is the  $L^2$  inner product on the interval  $J$ , and  $\phi \in H^{-1}$  is taken in a certain way so that  $\phi \notin \text{Im} D_u(L - F)(\lambda_0, u_0)$ . Weber proved that the equation  $H = 0$  is an isolated solution  $w_0 := (u_0, r_0, s_0, \lambda_0, 0) \in X_0$  if  $(\lambda_0, u_0) \in \Lambda \times H_0^1$  is a simple bifurcation point of the equation  $L - F = 0$ .

Now, we rewrite the equation  $H = 0$  as a fixed point problem. Since  $L|_{H_0^1} : H_0^1 \rightarrow H^{-1}$  is an isomorphism, we define  $\Phi \in \mathcal{L}(H^{-1}, H_0^1)$  by  $\Phi := (L|_{H_0^1})^{-1}$ . It is easy to show that there exists a positive constant  $C_1$  such that

$$(2.11) \quad \|\Phi f\|_{H^2} \leq C_1 \|f\|_{L^2}$$

for any  $f \in L^2$ . Note that in this case the constant  $C_1$  is easily determined, that is,  $C_1$  is available in numerical verification procedure.

We define the operator  $G : X_0 \rightarrow X$  by

$$(2.12) \quad G(u, r, s, \lambda, \mu) := \begin{pmatrix} \Phi(F(\lambda, u) - \mu\phi) \\ \Phi(D_u F(\lambda, u)r) \\ \Phi(D_u F(\lambda, u)s + D_\lambda F(\lambda, u)) \\ \lambda + (r, r) - 1 \\ \mu + (r, s) \end{pmatrix}.$$

Then, the equation  $H = 0$  is now equivalent to the fixed point problem

$$(2.13) \quad (u, r, s, \lambda, \mu) = G(u, r, s, \lambda, \mu).$$

We try to modify the usual procedure of numerical verification to the fixed point problem (2.13).

### 3. Formulation of Numerical Verification.

Let  $S_h \subset H_0^1$  be a finite element space. The projection  $P_{h0} : H_0^1 \rightarrow S_h$  is defined by

$$((u - P_{h0}u)', v_h') = 0, \quad \forall v_h \in S_h.$$

For  $S_h$ , we suppose the following assumption:

**Assumption 3.1.** *There exists a computable constant  $C_2$  which is independent of  $h$  and  $u$ , and satisfies the following estimate:*

$$(3.1) \quad \|u - P_{h0}u\|_{H_0^1} \leq C_2 h |u|_{H^2}, \quad \forall u \in H_0^1 \cap H^2. \quad \triangleleft$$

It is well known that the finite element space of piecewise linear functions satisfies Assumption 3.1.

Let  $X_h = Y_h := (S_h)^3 \times \mathbb{R}^2$ . The projection  $P_h : X \rightarrow X_h$  is defined by

$$(3.2) \quad P_h(u, r, s, \alpha, \beta) := (P_{h0}u, P_{h0}r, P_{h0}s, \alpha, \beta), \quad \text{for } (u, r, s, \alpha, \beta) \in X.$$

Suppose now that we are dealing with a simple bifurcation point. A finite element solution of the equation  $H = 0$  could be defined by

$$(3.3) \quad \begin{aligned} (u'_h, v'_{1h}) - (f(\lambda_h, x, u_h) + \mu_h \phi, v_{1h}) &= 0, \\ (r'_h, v'_{2h}) - (f_y(\lambda_h, x, u_h) r_h, v_{2h}) &= 0, \\ (s'_h, v'_{3h}) - (f_y(\lambda_h, x, u_h) s_h - f_\lambda(\lambda_h, x, u_h, v_{3h}) &= 0, \\ (r_h, r_h) - 1 &= 0, \\ (r_h, s_h) &= 0, \end{aligned}$$

for any  $(v_{1h}, v_{2h}, v_{3h}) \in (S_h)^3$ . As is stated in Section 1, however, bifurcation may be destroyed by discretization. Therefore, in general, a finite element solution of a simple bifurcation point does not exist.

Hence we just take  $w_h := (u_h, r_h, s_h, \lambda_h, \mu_h) \in X_h$  which is “close enough” to the solution  $w_0 \in X_0$  of the equation  $H = 0$ . In practical computation we try to make the absolute values of the left-hand side of (3.3) as small as possible. We then assume the following.

**Assumption 3.2.** *Let  $w_h := (u_h, r_h, s_h, \lambda_h, \mu_h) \in X_h$  be “finite element solution” computed by the above manner. We assume that the restricted operator  $P_h(I - DG(w_h))|_{X_h}$  has the inverse*

$$[I - DG^h]_h^{-1} : X_h \rightarrow X_h. \quad \triangleleft$$



If our "finite element solution" is close enough to the simple bifurcation point, Assumption 3.2 is satisfied.

In the sequel, we denote  $DG(w_h)$  and  $DF(w_h)$  by  $DG^h$  and  $DF^h$ , respectively.

Next, we introduce notions of *rounding* and *rounding error*. Let  $\epsilon$ , ( $0 < \epsilon < 1$ ) be a parameter. We first define the operator  $T_\epsilon : X_0 \rightarrow X$  by

$$(3.4) \quad T_\epsilon := I - ([I - DG^h]_h^{-1} P_h + \epsilon I)(I - G).$$

Note that if  $[I - DG^h]_h^{-1} P_h + \epsilon I$  has an inverse operator, the two fixed point equations  $w = G(w)$  and  $w = T_\epsilon(w)$  are equivalent. Our main tool of numerical verification has been the following fixed point theorem (for instance, see [Z]):

**Theorem 3.3 (Sadovskii's Fixed Point Theorem).** *Let  $X$  be a Banach space and  $U \subset X$  a nonempty, bounded, convex, closed subset. Suppose that the nonlinear operator  $T : U \rightarrow U$  is a condensing map. Then, there exists a fixed point  $u \in U$  of  $T$ :*

$$\exists u \in U \quad \text{such that} \quad u = Tu. \quad \triangleleft$$

Since  $T_\epsilon$  can be rewritten as

$$T_\epsilon = (1 - \epsilon)I + [I - DG^h]_h^{-1} P_h(I - G) + \epsilon G,$$

$T_\epsilon$  is a condensing map from  $X_0$  to  $X$ . Hence, if we have a nonempty, bounded, convex, closed subset  $U \subset X_0$  such that  $T_\epsilon U \subseteq U$ , we can conclude that there exists a fixed point of  $T_\epsilon$ . Moreover, if  $[I - DG^h]_h^{-1} + \epsilon I$  is invertible, the fixed point of  $T_\epsilon$  is a solution of the equation  $H = 0$ . Hence, our verification is reduced to the construction of such  $U$  on the memory of computer.

The approximations of an element  $u \in H_0^1$ , a subset  $U \subset H_0^1$ , and operators defined on  $H_0^1$  in a certain finite element space  $S_h$  are called their **rounding**. The error of the rounding is called **rounding error**. These notions are defined by projection.

The rounding  $\tilde{T}_\epsilon$  of  $T_\epsilon$  is defined by  $\tilde{T}_\epsilon := P_h \circ T_\epsilon$ , where  $P_h$  is the projection defined by (3.2). Then, we see that

$$(3.5) \quad \tilde{T}_\epsilon = \tilde{I} - ([I - DG^h]_h^{-1} + \epsilon \tilde{I})(\tilde{I} - \tilde{G}),$$

where  $\tilde{I} := P_h \circ I$  and  $\tilde{G} := P_h \circ G$ . Let  $U \subset X$ . The rounding  $R(T_\epsilon U)$  is defined as the image of  $\tilde{T}_\epsilon$ :

$$(3.6) \quad R(T_\epsilon U) := \{w_h \in X_h \mid w_h = \tilde{T}_\epsilon(w), w \in U\}.$$

We define the rounding error  $RE(T_\epsilon U)$  by

$$\alpha := \sup_{w \in U} \|T_\epsilon(w) - \tilde{T}_\epsilon(w)\|_X,$$

$$C := C_1 C_2, \quad (C_1, C_2 \text{ are defined by (2.11), (3.1), respectively.}),$$

$$RE(T_\epsilon U) := \left\{ (\psi_1, \psi_2, \psi_3) \in (S_h^\perp)^3 \mid \max_i \|\psi_i\|_{H_0^1} \leq \alpha, \max_i \|\psi_i\|_{L^2} \leq Ch\alpha \right\} \times \{(0, 0)\}.$$

Note that  $RE(T_\epsilon U)$  is a subset of  $(H_0^1)^3 \times \{(0, 0)\}$ . Then, as in [TN] and [WN], we have

**Theorem 3.4.** *Let  $U \subset X_0$  be a nonempty, bounded, convex, closed subset. If*

$$(3.7) \quad R(T_0 U) \oplus RE(T_0 U) \overset{\circ}{\subset} U,$$

*then, there exists a solution  $w \in U$  of the fixed point problem  $w = G(w)$ . Here,  $A \overset{\circ}{\subset} B$  means  $\text{closure}(A) \subset \text{interior}(B)$ .*

#### 4. Numerical Verification.

By Theorem 3.4, in the set  $U \subseteq X_0$  which satisfies (3.7), there exists at least one solution  $w \in X_0$  of the fixed point problem  $w = G(w)$ . Therefore, if we construct such  $U$  on the memory of computer, the solution of the fixed point problem is said to verified numerically. This is what we shall do in this section.

Let  $\{\phi_j\}_{j=1}^M$  be the basis of  $(S_h)^3$ . Let  $\Theta_h$  be the set of linear combinations of intervals and  $\phi_j$ :

$$(4.1) \quad \Theta_h := \left\{ \left( \sum_{j=1}^M A_j \phi_j, A_{M+1}, A_{M+2} \right) \mid A_j \subset \mathbb{R} \text{ are intervals} \right\}.$$

That is, an element  $\omega \in \Theta_h$  is the set

$$\omega = \left( \sum_{j=1}^M A_j \phi_j, A_{M+1}, A_{M+2} \right) := \left\{ \left( \sum_{j=1}^M a_j \phi_j, a_{M+1}, a_{M+2} \right) \mid a_j \in A_j \right\}.$$

Let  $\mathbb{R}^+$  be the set of nonnegative reals. For  $\alpha \in \mathbb{R}^+$ , we define the set  $[\alpha] \subset (S_h^\perp)^3 \times \{(0, 0)\} \subset (H_0^1)^3 \times \{(0, 0)\}$  by

$$(4.2) \quad [\alpha] := \left\{ (\psi_1, \psi_2, \psi_3) \in (S_h^\perp)^3 \mid \max_i \|\psi_i\|_{H_0^1} \leq \alpha, \max_i \|\psi_i\|_{L^2} \leq Ch\alpha \right\} \times \{(0, 0)\}.$$

We define the following iteration:

**Definition 4.1.** Let  $w_h \in X_h$  be the "finite element solution."

(1) We set  $\Delta w_h^0 := \{w_h\}$  and  $\alpha_0 := 0$  as the initial values.

(2) For  $n \geq 1$ , we define  $U^{n-1} \subset X_0$ ,  $\Delta w_h^n \subset X_h$ , and  $\alpha_n \in \mathbb{R}^+$  inductively by

$$(4.3) \quad \begin{cases} U^{n-1} := \Delta w_h^{n-1} + [\alpha_{n-1}], \\ \Delta w_h^0 := \tilde{T}_0 U^{n-1}, \\ \alpha_n := Ch \sup_{y \in U^{n-1}} \|\tilde{G}(y)\|, \end{cases}$$

where  $\|\tilde{G}(y)\|$  is defined, for  $y := (y_1, y_2, y_3, a, b) \in X$ , by

$$\|\tilde{G}(y)\| := \max \left\{ \|f(a, x, y_1) - b\phi\|_{L^2}, \|f_y(a, x, y_1)y_2\|_{L^2}, \|f_y(a, x, y_1)y_3 + f_\lambda(a, x, y_1)\|_{L^2} \right\}. \quad \triangleleft$$

Note that it is very difficult or impossible to estimate  $\Delta w_h^n$  and  $\alpha_n$  in (4.3) exactly. It is, however, possible and easy to enclose each coefficient interval by a slightly bigger interval, that is, overestimate them (cf. [WN]).

Now, let  $\delta > 0$  be a small real. We define

$$(4.4) \quad \begin{cases} \Delta \tilde{w}_h^n := \Delta w_h^n + \left( \sum_{j=1}^M [-1, 1]\delta\phi_j, [-1, 1]\delta, [-1, 1]\delta \right), \\ \tilde{\alpha}_n := \alpha_n + \delta. \end{cases}$$

The definition of (4.4) is called  $\delta$ -extension. Let  $\tilde{U} := \Delta \tilde{w}_h^n + [\tilde{\alpha}_n]$ . Let  $\Delta \bar{w}_h \subset X_h$  and  $\bar{\alpha}_n \in \mathbb{R}^+$  be obtained by the iteration (4.3) from  $\tilde{U}$ :

$$(4.5) \quad \begin{cases} \Delta \bar{w}_h := \tilde{T}_0 \tilde{U}, \\ \bar{\alpha}_n := Ch \sup_{y \in \tilde{U}} \|\tilde{G}(y)\|. \end{cases}$$

For these sets, the inclusion  $\Delta \bar{w}_h \overset{\circ}{\subset} \Delta \tilde{w}_h^n$  is defined by  $B_j \overset{\circ}{\subset} A_j$  ( $j = 1, \dots, M+2$ ), where  $\Delta \tilde{w}_h^n = \left( \sum_{j=1}^M A_j \phi_j, A_{M+1}, A_{M+2} \right)$  and  $\Delta \bar{w}_h = \left( \sum_{j=1}^M B_j \phi_j, B_{M+1}, B_{M+2} \right)$ .

To judge whether or not  $\tilde{U}$  is what we want, we have the following theorem:

**Theorem 4.2.** If we find

$$(4.6) \quad \begin{cases} \Delta \bar{w}_h \overset{\circ}{\subset} \Delta \tilde{w}_h^n, \\ \bar{\alpha}_n < \tilde{\alpha}_n, \end{cases}$$

we conclude that there exists a solution  $w \in \tilde{U}$  of the fixed point problem  $w = G(w)$ .

## 5. A Numerical Example.

In this section we present an example of numerical verification for the following equation:

$$(5.1) \quad \begin{cases} -u'' = \lambda u + \sin(2\pi x), & \text{in } J, \\ u(0) = u(1) = 0. \end{cases}$$

It is easy to check that  $(\pi^2, \sin(2\pi x)/(3\pi^2))$  is a simple bifurcation point of (5.1). We set  $\phi(x) := \sin(\pi x)$ . We consider the extended equation

$$(5.2) \quad \begin{cases} -u'' = \lambda u + \sin(2\pi x) + \mu\phi, \\ -r'' = \lambda r, \\ -s'' = \lambda s + u, \\ (r, r) - 1 = 0, \quad (r, s) = 0 \\ u(0) = u(1) = r(0) = r(1) = s(0) = s(1) = 0. \end{cases}$$

By a simple calculation we find that the equation (5.2) has an isolated solution  $w_0 := (u_0, r_0, s_0, \lambda_0, 0) \in X_0$ , where  $u_0 := \sin(2\pi x)/(3\pi^2)$ ,  $r_0 := \sqrt{2}\sin(\pi x)$ ,  $s_0 := \sin(2\pi x)/(9\pi^4)$ , and  $\lambda_0 := \pi^2$ .

Let  $N := 100$ . We divide  $J$  equally into  $N$  small intervals. Let  $S_h$  the finite element space of piecewise linear functions.

We try to verify the solution  $w_0$ . The following are the result of verification. We show  $\tilde{\alpha}_n$  and the constructed set  $\tilde{U} = (\sum_{j=1}^{297} A_j \phi_j, A_{298}, A_{299})$ , where  $A_j := [a_j, b_j]$ .

The iteration number = 5,

$$\tilde{\alpha}_n = 3.147062\text{D-}2,$$

$$\lambda_h = 9.87042 \in A_{298} = (9.86224, 9.87860) \text{ and } |A_{298}| = 1.45585\text{D-}2,$$

$$\mu = 4.80\text{D-}15 \in A_{299} = (-1.03775\text{D-}2, 1.03775\text{D-}2) \text{ and } |A_{299}| = 2.07550\text{D-}2,$$

$$\max_{1 \leq i \leq 99} |A_i| = 2.29400\text{D-}2,$$

$$\max_{100 \leq i \leq 198} |A_i| = 9.01306\text{D-}4,$$

$$\max_{199 \leq i \leq 297} |A_i| = 9.95609\text{D-}4,$$

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